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TECHNICAL MEMORANDUM

ACCURATE APPROXIMATE SOLUTIONS TO OSCILLATORY PROBLEMS WITH PERTURBING SINGULAR DAMPING

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ABSTRACT

An attempt is being made to obtain approximate solutions of improved accuracy for a class of differential equations of the form

$$\frac{d^2y}{dx^2} + \varepsilon\mu(x)\frac{dy}{dx} + \omega_c^2 y = 0,$$

where ε is a real parameter less than unity, ω_c is a positive real constant of order unity, and $\mu(x)$ is a singular function of x in the region of interest.

It does not appear to be possible to obtain a general analytic expression for the error estimate of the approximate solution. For the case $\mu(x) = x^{-2}$, however, it is shown that the approximate solution is accurate to $0(\varepsilon^2)$, as $x \to 0^-$ from negative values, by comparing it with the numerically integrated solution. For the same case, the approximate solution is orders of magnitude more accurate than Poincaré's first order perturbation solution, which is accurate to $0(\varepsilon^2 \ln|x|/|x|)$ as $x \to 0^-$. The approximate analytic solution of the present method can yield significant mathematical and physical insight to actual problems.

This work arose in search for analytic solutions to a linearized form of the restricted three-body problem.

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SUBJECT: Accurate Approximate Solutions to Oscillatory Problems with Perturbing Singular Damping Case 103-7

DATE: November 30, 1970

FROM: C.C.H. Tang

TM-70-1011-8

TECHNICAL MEMORANDUM

Introduction

Many physical problems are characterized by the presence of a perturbing force which can be either constant or varying. The exact solution of a simple linear oscillator with constant damping is well known, but that with arbitrarily varying damping is unobtainable without resorting to numerical integration. For example, the following differential equation for a linear oscillator with a perturbing singular damping term cannot be solved exactly by presently known methods and functions

$$\frac{d^2y}{dx^2} + \varepsilon\frac{1}{x^2}\frac{dy}{dx} + \omega_c^2 y = 0 \qquad (1)$$

where ε is a positive real parameter much less than unity and ω_c^2 a positive real constant of $0(1)$. It will be shown that for $|x| > \sqrt{\varepsilon}$ Poincaré's perturbation method of small parameter expansion will yield a first order homogeneous solution accurate to $0\left(\varepsilon^2 \frac{\ln x}{x}\right)$ which is singular as $x \to 0$. For $0 < |x| < \sqrt{\varepsilon}$ higher order perturbation solution cannot improve the accuracy of the solution in the neighborhood of the singular point at x=0, because of the singular nature of the perturbing term in Eq. (1).

It is the attempt of this paper to show a new method of obtaining approximate solutions for a class of second order

ordinary linear differential equations of the form (prime denotes derivative)

$$y'' + \varepsilon\mu(x)y' + \omega_c^2 y = 0 . \qquad (2)$$

For the case $\mu(x) = x^{-2}$ it is shown that the approximate solution has an error of $0(\varepsilon^2)$ as $x \to 0^-$. This error is obtained by comparing the approximate solution with the exact solution obtained by numerically integrating Eq. (1). It has been shown in a previous paper[1] that when the coefficients $\varepsilon\mu(x)$ and ω_c^2 are related in certain specific ways, Eq. (2) can be solved exactly by means of elementary functions in finite terms. In fact, if $\varepsilon\mu(x) = \omega_c^2 x$ or $\varepsilon\mu(x) = 2c\,\frac{e^{cx}-ke^{-cx}}{e^{cx}+ke^{-cx}}$, where c and k are two arbitrary constants, the coefficients $\varepsilon\mu(x)$ and ω_c^2 can be shown to be related.

To solve Eq. (2) in a general form, we first transform it into a first order nonlinear differential equation of the Riccati form and then solve the nonlinear equation approximately. The accuracy of the approximate solution thus obtained for the case $\mu(x) = x^{-2}$ is orders of magnitude better than that obtained by Poincaré's first order small parameter expansion method in a region where $x \to 0^-$ from negative values.

Formulation and Error Estimate

For simplicity we shall assume that $\mu(x)$ is real and analytic except at a finite number of poles. In an attempt to obtain the general solution of Eq. (2), we let one of its two solutions take the following form[1]:

$$y_1 = A_0 e^{\int_{x_i}^{x} [Z(x) - \frac{1}{2}\varepsilon\mu(x)]dx}, \tag{3}$$

and the other solution becomes

$$y_2 = B_0 e^{\int_{x_i}^{x} (Z - \frac{1}{2}\varepsilon\mu)dx} \int_{x_i}^{x} e^{-2\int_{x_i}^{x} Zdx} dx, \tag{4}$$

where A_0 and B_0 are constants to be determined by the initial conditions at x_i.

Substituting $y = y_1 + y_2$ into Eq. (2), we obtain

$$(Z' + Z^2 + \omega_c^2 + \frac{1}{2}\varepsilon\mu' - \frac{1}{4}\varepsilon^2\mu^2)y_1 = 0. \tag{5}$$

Excluding the trivial solution $y_1=0$, we have the condition for obtaining the solution of Eq. (2) as:

$$Z' + Z^2 = -\omega_c^2 + \frac{1}{2}\varepsilon\mu' + \frac{1}{4}\varepsilon^2\mu^2, \tag{6}$$

which is recognized as a nonlinear ordinary differential equation in canonical Riccati form. In fact it is well-known[1][2] that any second order ordinary linear differential equation can be transformed into this form.

In an effort to solve Eq. (6) in terms of $\mu(x)$, we first try the apparently simplest approximate solution in the form

$$Z_a = a + b\mu, \tag{7}$$

where a and b are constants to be determined.

Differentiating and squaring Eq. (7), we obtain

$$y_1(Z'_a + Z_a^2) = (b\mu' + a^2 + 2ab\mu + b^2\mu^2)y_1. \tag{8}$$

Comparison of Eqs. (5) and (8) yields the constants

$$a = \pm i\omega_c \qquad \text{and} \qquad b = \frac{1}{2}\varepsilon \ . \tag{9}$$

Thus $$y_1(Z'_a + Z_a^2) = y_1(-\omega_c^2 + \frac{1}{2}\varepsilon\mu' + \frac{1}{4}\varepsilon^2\mu^2) \pm E, \tag{10}$$

where

$$E = \pm i\omega_c \varepsilon \mu y_1 \tag{10a}$$

is an error term which appears in Eq. (2) because of the approximate nature of Eq. (7) as a solution of Eq. (6) or (2). It is important to note that the error term as shown in Eq. (10a) is imaginary and is simply proportional to the product of the perturbing damping term and the undamped frequency of the original differential equation (2). The significance of this error term being imaginary will be exploited later. Substituting Eq. (9) into Eqs. (7), (3) and (4), we obtain the approximate solution of Eq. (2) as

$$y_0 = e^{i\omega_c x}\left(A_0 + B_0 \int_{x_i}^{x} e^{-i2\omega_c x}\, e^{-\varepsilon \int_{x_i}^{x} \mu dx}\, dx\right) = A_0 y_1 + B_0 y_2 \ . \tag{11}$$

It turns out that Eq. (11) as an approximate solution to Eq. (2) has a very simple intuitive interpretation. The first solution y_1 is simply a solution, in complex form, of Eq. (2) with the

damping term neglected and the second solution y_2 is obtained by employing y_1 as if y_1 were the exact solution of Eq. (2). A measure of the accuracy of Eq. (11) can be first tested by two extreme cases; namely, (i) when ε or $\mu(x)$ approaches zero, Eq. (11) reduces to the exact solution of a linear oscillator, (ii) when ω_c approaches zero, Eq. (11) reduces to the exact solution of $y'' + \varepsilon\mu(x)y' = 0$. Accordingly the solution in the form of Eq. (11) is expected to be very accurate even in the middle region (composite region) where $\varepsilon\mu=\omega_c$ and an example to be shown later indicates that this is indeed true.

Since, in general, we have no way of obtaining the exact solution of Eq. (2), it is very difficult to obtain even the error bound of the approximate solution in analytic form. The numerical accuracy of Eq. (11) can be obtained for any particular $\mu(x)$, however, by comparing it with the solution achieved by numerically integrating the differential equation (2). It is reasonable to assume that the error term $\pm i\omega_c\varepsilon\mu_1$ of Eq. (10a) might give us an indirect measure of the accuracy of the approximate solution y_o of Eq. (11). The ratio of the error term to the exact $(Z'+Z^2)$ provides relative comparison of the accuracy of the approximate solution in the region of interest where $\mu(x)$ approaches a singularity. Since the error term is imaginary and the exact $(Z'+Z^2)$ is real due to the specification that ω_c, ε, and $\mu(x)$ are all real quantities, the relative error can be expressed as:

$$E_r = \frac{\left[(Z'+Z^2)^2 + (\omega_c \varepsilon\mu)^2\right]^{\frac{1}{2}} - (Z' + Z^2)}{Z' + Z^2} = \left[1 + \frac{\omega_c \varepsilon\mu}{Z'+Z^2}^2\right]^{\frac{1}{2}} - 1$$

$$\doteq \frac{1}{2}\left(\frac{\omega_c \varepsilon\mu}{Z' + Z^2}\right)^2 = \frac{1}{2}\left(\frac{\omega_c \varepsilon\mu}{-\omega_c^2 + \frac{1}{2}\varepsilon\mu' + \frac{1}{4}\varepsilon^2\mu^2}\right)^2 . \qquad (12)$$

Equation (12) assumes the following simple forms:

$$E_r \doteq \left(\frac{\varepsilon\mu}{\omega_c}\right)^2 \qquad \text{when } \omega_c^2 >> \varepsilon^2\mu^2 \text{ and } \varepsilon\mu', \qquad (13a)$$

$$E_r \doteq \left(\frac{\omega_c}{\varepsilon\mu}\right)^2 \qquad \text{when } \varepsilon^2\mu^2 >> \omega_c^2 \text{ and } \varepsilon\mu', \qquad (13b)$$

We note from Eqs.(13) that the errors in two separated regions centered around x_1 and x_2 could be of the same order of magnitude when, for example, $\frac{\varepsilon\mu(x_1)}{\omega_c} \doteq \frac{\omega_c}{\varepsilon\mu(x_2)}$. Fulfillment of such a condition may lead to the fact that errors could remain fairly constant over the region between x_1 and x_2. This will be shown to be true in a numerical example.

Another indirect error magnitude estimate can be made by comparing the error term $E = i\omega_c \varepsilon\mu y_1$ of Eq. (10a) with that obtained by Poincaré's perturbation method. The first order perturbation

solution of Eq. (2) has an error of $0(\varepsilon^2 y_{2nd})$ and has the form

$$y = y_{0th} + \varepsilon y_{1st} = y_{0th} + \frac{\varepsilon}{\omega_c} \int_{xi}^{x} \mu(\xi) y'_{0th}(\xi) \text{ Sin } \omega_c(x-\xi)d\xi, \tag{14}$$

where y_{0th}, y_{1st} and y_{2nd} are, respectively, the zeroth, first, and second order solutions, and $y_{0th} = C\text{Sin } \omega_c(x-x_0)$ with C and x_0 as constants.

Substituting Eq. (14) into the differential equation (2) yields an error term (due to the first order perturbation solution):

$$E_p = -\varepsilon^2 \mu y'_{1st} = \varepsilon^2 \mu \omega_c \left[\int_{x_i}^{x} \mu y'_{0th} dx + \omega_c \int_{x_i}^{x} \left(\int_{\xi_i}^{\xi} \mu(z) y'_{0th}(z) dz \right) \right.$$
$$\left. \text{Sin } \omega_c(x-\xi)d\xi \right]. \tag{15}$$

Eq. (15) is a real quantity since the terms involved are specified to be real. To compare the error term of Eq. (10a) with that of Eq. (15), it will be convenient to specify the general functional form of the singular coefficient $\mu(x)$. Let $\mu(x) = \frac{1}{x^n}$ and x vary in the region where $0 < |x| \leq \varepsilon$, then from Eq. (10a) we have, as x approaches the singular point x=0,

$$E = i\omega_c \varepsilon \mu e^{i\omega_c x} \doteq i\frac{\omega_c \varepsilon}{x^n} ,$$

where n is an integer.

From Eq. (15) we have for the first order perturbation solution,

$$E_{p1} \doteq \frac{\omega_c \varepsilon^2}{x^n}\left[0\left(\frac{\ln x}{x^{n-1}}\right) + 0\left(\frac{\ln x}{x^{n-2}}\right)\right] \doteq 0\left(\frac{\omega_c \varepsilon^2 \ln x}{x^{2n-1}}\right) . \qquad (16)$$

It can be shown that for the k^{th} order perturbation solution

$$E_{pk} \doteq \frac{\omega_c \varepsilon^{k+1}}{x^n}\left[0\left(\frac{\ln x}{x^{n+k-2}}\right) + 0\left(\frac{\ln x}{x^{n+k-3}}\right)\right] \doteq 0\left(\frac{\omega_c \varepsilon^{k+1} \ln x}{x^{2n+k-2}}\right) . \qquad (16a)$$

In the region where $0 << |x| \leq \varepsilon$, E is always smaller than E_{pk}. Accordingly it appears to be that in some cases the present approximate solution is more accurate than Poincaré's perturbation solution as $\mu(x)$ approaches its apparent singular point. Using the same relative error criterion used in Eq. (12), we obtain for the perturbation method

$$E_{pk_r} = 0\left(\frac{\omega_c \varepsilon^{k+1} \ln x}{x^{2n+k-2}}\right) \Big/ (z' + z^2) \qquad (17)$$

Comparison of Eqs. (12) and (17) indicates that the region in which the accuracy of the present approximate solution is better than that of the perturbation solution can be extended from the region $0 < |x| \leq \varepsilon$ to the region where $|x|$ is slightly $> \varepsilon$.

At this point it is appropriate to ask the question what is the physical meaning and significance of the error term $i\omega_c \varepsilon \mu y_1$ being imaginary? It is evident that if we had been able to find the exact solution of Eq. (6), the exact $(Z' + Z^2)$ would be pure real due to the fact that ω_c, ε, and μ are all specified to be real quantities. The presence of the pure imaginary error term thus gives a measure of error in Z_a. However, the percentage error E_r in terms of the exact $(Z' + Z^2)$ for the case of a small imaginary error term should be much less than that for the case of a real error term of the same magnitude. In fact we have for the real error term case $E_r = \frac{\omega_c \varepsilon \mu}{Z' + Z^2}$, which is usually much less than unity and is accordingly orders of magnitude larger than $\frac{1}{2}\left(\frac{\omega_c \varepsilon \mu}{Z' + Z^2}\right)^2$ of Eq. (12). This is why it is important to have a small error term which is imaginary.

Having conceptually understood the problem, we are now in a position to be able to improve the accuracy of the approximate solution by letting $Z_a = Z_{ar} + iZ_{ai}$ and then adjusting Z_{ar} and Z_{ai} such that the magnitude $\left|(Z_{ar}^2 + Z_{ai}^2)^{1/2}\right|$ may approach closer to the exact Z. When $\left|(Z_{ar}^2 + Z_{ai}^2)^{1/2}\right| = Z$, we have the exact solution together with the phase information $\theta = \tan^{-1}\frac{Z_{ai}}{Z_{ar}}$ which is the phase delay or advance with respect to the unperturbed linear oscillator case. To state the problem more precisely, we should adjust Z_{ar} and Z_{ai} in such a way that the magnitude of $(Z_a' + Z_a^2)$ approaches that of $Z' + Z^2 = -\omega_c^2 + \frac{1}{2}\varepsilon\mu' + \frac{1}{4}\varepsilon^2\mu^2$, i.e., $Z_a' + Z_a^2$ should lie as close as possible (a) to the circle of radius equal to $(Z' + Z^2)$ and (b) to the real axis $(Z' + Z^2)$.

According to the arguments of the preceding paragraph we can improve the solution accuracy of Eq. (11), i.e., Eq. (7), by reasoning that the constants a and b in Eq. (7) should have such values that Eq. (8) would have, in addition to the imaginary part of the error term, a real part of the error term with a sign opposite to $(Z' + Z^2)$ and with a magnitude much smaller than the imaginary part. Let both constants a and b be complex, then

$$Z_a = (a_r + ia_i) + (b_r + ib_i)\mu \ . \tag{18}$$

Differentiating and squaring Eq. (18) yields

$$\begin{aligned} Z_a' + Z_a^2 = {} & (b_r + ib_i)\mu' + (a_r^2 - a_i^2) + i\ 2a_r a_i \\ & + 2[(a_r b_r - a_i b_i) + i(a_r b_i + a_i b_r)]\mu \\ & + [(b_r^2 - b_i^2) + i\ 2b_r b_i]\mu^2 . \end{aligned} \tag{19}$$

Comparing Eq. (19) with Eq. (6), we have the following equations

$$\left.\begin{aligned} & a_r^2 - a_i^2 = \omega_c^2 \ , \\ & (b_r + ib_i)\mu' = \tfrac{1}{2}\,\varepsilon\mu' \ , \\ & (b_r^2 + b_i^2 + i\ 2b_r b_i)\mu^2 = \tfrac{1}{4}\,\varepsilon^2\mu^2 \ , \\ & i\ 2a_r a_i = 0 \ , \\ & 2[(a_r b_r - a_i b_i) + i(a_r b_i + a_i b_r)]\mu = 0 \ . \end{aligned}\right\} \tag{20}$$

The system of Eqs. (20) is overdetermined since there are only four constants to be determined by eight equations. Guided by the reasoning mentioned in the preceding paragraph, we can choose the constants by reducing or minimizing the error magnitude in Eq. (10) in the region of interest. The simplest and best choice of these constants appears to be:

$$\begin{aligned} a_r &= 0 \quad , \quad a_i = \pm\omega_c \quad , \\ b_r &= \frac{\varepsilon}{2} \quad , \quad b_i = k\varepsilon b_r \quad , \end{aligned} \tag{21}$$

where k is an adjusting parameter, varying in the range approximately between 0 and $+1$, to be explained later. Substituting Eq. (21) into Eq. (10), we obtain

$$\begin{aligned} Z_a' + Z_a^2 &= (-\omega_c^2 + \tfrac{1}{2}\varepsilon\mu' + \tfrac{1}{4}\varepsilon^2\mu^2) - k\varepsilon^2\mu(\omega_c + \tfrac{1}{4}k\varepsilon^2\mu) \\ &\quad + \left[i\,\omega_c\varepsilon\mu + \tfrac{1}{2}\,k\varepsilon^2(\varepsilon\mu^2 + \mu') \right] \\ &= (Z' + Z^2) - F_r + iF_i, \end{aligned} \tag{22}$$

where

$$F_r = k\varepsilon^2\mu(\omega_c + \tfrac{1}{4}\,k\varepsilon^2\mu) \text{ and } F_i = \omega_c\varepsilon\mu + \tfrac{1}{2}\,k\varepsilon^2(\varepsilon\mu^2 + \mu'). \tag{22a}$$

Equation (22) reduces to Eq. (10) when $b_i = 0$. Note that the real part of the error is in fact much smaller than the imaginary part of the error by a factor of about ε. The relative error then is

$$E_r = \frac{\left[(Z' + Z^2 - F_r)^2 + F_i^2\right]^{\frac{1}{2}} - (Z' + Z^2)}{Z' + Z^2} \tag{23}$$

$$= \frac{F_r^2 + F_i^2}{2(Z' + Z^2)^2} - \frac{F_r}{Z' + Z^2} = \frac{1}{2(Z' + Z^2)^2}\left[F_r^2 + F_i^2 - 2F_r(Z' + Z^2)\right]$$

where both F_r and F_i are functions of the parameters k and ε. Equation (23) reduces to Eq. (12) when k=0, i.e., when b_i=0. It is evident from Eq. (23) that the relative error E_r will vanish if $F_r^2 + F_i^2 = 2F_r(Z' + Z^2)$ at certain point x_e. This implies that an exact solution can be obtained at the point x_e, around which the error of the approximate solution changes sign. For a prescribed x_e where an exact solution is most desired, there is always a corresponding value of the parameter k which can be obtained by setting Eq. (23) equal to zero: i.e.,

$$F_r^2 + F_i^2 - 2F_r(Z'+Z^2) \doteq (\omega_c\varepsilon\mu)^2 + (\tfrac{1}{2}k\varepsilon\mu')^2 + k\varepsilon^2\mu\left\{2\omega_c^3 + \frac{1}{2}\left[3\omega_c^2\varepsilon + (1-\tfrac{1}{2}\varepsilon^2)\mu'\right]k\varepsilon\mu + \frac{1}{2}(1+k^2\varepsilon^2)\omega_c\varepsilon^2\mu^2 + \frac{1}{4}\left[1+\tfrac{1}{4}\varepsilon^2(k^2-2)\right]k\varepsilon^2\mu^3\right\} = 0 . \tag{24}$$

Since ε is much less than unity and ω_c close to unity Eq. (24) can be simplified as

$$(\omega_c\varepsilon\mu)^2 + (\tfrac{1}{2}k\varepsilon\mu')^2 + k\varepsilon^2\mu\left[2\omega_c^3 + \frac{1}{2}k\varepsilon\mu'\mu + \frac{1}{2}\omega_c\varepsilon^2\mu^2 + \frac{1}{4}k\varepsilon^2\mu^3\right] = 0 . \tag{25}$$

For a given $\mu(x_e)$, Eq. (25) is a quadratic equation in k.

Now substituting Eq. (21) into Eqs. (7), (3), and (4), we obtain the general approximate solution of Eq. (2) as

$$y_k = e^{i\omega_c x + \frac{1}{2}k\varepsilon^2\int_{x_i}^{x}\mu dx}\left[A_k + B_k\int_{x_i}^{x} e^{-i\left(2\omega_c x + k\varepsilon^2\int_{x_i}^{x}\mu dx\right)} e^{-\varepsilon\int_{x_i}^{x}\mu dx}\, dx\right] \tag{26}$$

which reduces to Eq. (11) by setting k=0.

Note that Eq. (26) is written in such a way that we require the coefficient of the dominant term of the function $\mu(x)$ to be positive unity, i.e., the sign and the coefficient of the dominant term of $\mu(x)$ must be absorbed into ε.

In passing we mention that the constants in the over-determined system of Eq. (20) can also be optimally determined in the region of interest in the least squares sense whenever the functional form of $\mu(x)$ is given.

Comparison Examples

In the preceding section we have argued qualitatively that the approximate solution in the form of Eq. (26) should be more accurate than the perturbation solution of Eq. (2) in the region where there is a singular point. Here we attempt to show quantitatively the numerical accuracy of the solution of Eqs. (11) and (26) for a specific differential equation with $\mu = \frac{1}{x^2}$, i.e.,

$$y'' + \varepsilon\,\frac{1}{x^2}y' + \omega_c^2\, y = 0\ , \tag{27}$$

where we let $\omega_c = 1$ and $\varepsilon = 10^{-4}$. The approximate solution of Eq. (27) via Eqs. (11) and (26) becomes, respectively,

$$y_0 = e^{i\omega_c x}\left(A_0 + B_0\int_{x_i}^{x} e^{-i2\omega_c x}\cdot e^{\frac{\varepsilon}{x}}\,dx\right) = y_{k=0}\ , \tag{28}$$

or

$$y_k = e^{i\left(\omega_c x - \frac{k\varepsilon^2}{2x}\right)}\left[A_k + B_k\int_{x_i}^{x} e^{-i\left(2\omega_c x - \frac{k\varepsilon^2}{x}\right)}\cdot e^{\frac{\varepsilon}{x}}\,dx\right]\ , \tag{29}$$

where $A_k = y(x_i)e^{-i\left(\omega_c x_i - \frac{k\varepsilon^2}{2x_i}\right)}$, (29a)

$$B_k = \left[y'(x_i) - i\left(\omega_c + \frac{k\varepsilon^2}{2x_i^2}\right)y(x_i)\right]\ e^{\frac{\varepsilon}{x_i}}\ e^{i\left(\omega_c x_i - \frac{k\varepsilon^2}{2x_i}\right)}\ ,$$

and $y(x_i)$ and $y'(x_i)$ are prescribed initial conditions.

Note that Eq. (28) can be obtained from Eq. (29) by setting k=0. Comparison of solutions of Eqs. (28) and (29) and the 1st order Poincaré solution [Eq. (14)] with the numerically integrated solution of Eq. (27) should give us a clear picture of the accuracy

of the solutions in the form of Eq. (28) or (29). Before carrying out the actual numerical comparisons, we shall first obtain the Poincaré solution of Eq. (27). The formal Poincaré solution has the form

$$y = y_{0th} + \varepsilon y_{1st} + \varepsilon^2 y_{2nd} + \varepsilon^3 y_{3rd} + \ldots\ldots\ldots , \quad (30)$$

together with the initial boundary conditions

$$y_{1st}(x_i) = y_{2nd}(x_i) = y_{3rd}(x_i) = 0 \quad ,$$

$$y'_{1st}(x_i) = y'_{2nd}(x_i) = y'_{3rd}(x_i) = 0 \quad ,$$

where $y_k = \frac{1}{\omega_c} \int_{x_i}^{x} -\frac{1}{\xi^2}\, y'_{k-1}(\xi) \mathrm{Sin}\omega_c(x-\xi)d\xi$, k=1st, 2nd, 3rd, . . . (31)

$$y_{1st} = C - \frac{1}{2x_i}\left[\mathrm{Sin}\omega_c(x-x_0) + \mathrm{Sin}\omega_c(x+x_0-2x_i)\right]$$

$$+ \omega_c \mathrm{Sin}\omega_c(x+x_0) \int_{x_i}^{x} \frac{1}{x}\, \mathrm{Sin}2\omega_c x dx$$

$$+ \omega_c \mathrm{Cos}\omega_c(x+x_0) \int_{x_i}^{x} \frac{1}{x}\, \mathrm{Cos}2\omega_c x dx \doteq 0\,(\ln x) \quad , \quad (31a)$$

$$y_{2nd} = -\frac{1}{\omega_c}\int_{x_i}^{x}\frac{1}{\xi^2}\,y'_{1st}(\xi)\,\mathrm{Sin}\omega_c(x-\xi)\,d\xi \doteq O(\frac{\ln x}{x}) \quad , \qquad (31b)$$

$$y_{3rd} \doteq O(\frac{\ln x}{x^2}) \quad , \qquad y_{4th} \doteq O(\frac{\ln x}{x^3}) \quad , \ldots\ldots \qquad (31c)$$

It is evident that the explicit evaluation of y_{2nd} or higher order solution is extremely complex and usually impractical. Since it requires many lines to write down the expression for y_{2nd}, we indicate instead the order of magnitude as shown in Eq. (31b). Thus Eq. (30) becomes

$$y = C\mathrm{Sin}\omega_c(x-x_0) + \varepsilon O(\ln x) + \varepsilon^2 O(\frac{\ln x}{x}) + \varepsilon^3 O(\frac{\ln x}{x^2}) + \ldots \qquad (32)$$

Equation (32) indicates that the inclusion of a finite number of higher order solutions does not improve the accuracy of the solution when $|x| \leq \sqrt{\varepsilon}$. This is so because of the presence of the apparent singular perturbing term $\frac{\varepsilon}{x^2}y'$ in Eq. (27).

To carry out the actual numerical comparison, we use the first order Poincaré solution, i.e., the first two terms of Eq. (32) where we arbitrarily let $C=1$ and $x_0=-2.4$. Since we are interested in the behaviour of the solution near the singular point $x=0$, we let the integration limits vary from the initial limit $x=-0.3$ to the upper limit $x\to 0$. The initial boundary conditions for Eqs. (28) and (29) are

$$y(x_i) = C\sin\omega_c(x-x_0) = \text{Sin } 2.1$$

and

$$y'(x_i) = C\omega_c\cos\omega_c(x-x_0) = \text{Cos } 2.1$$

All computations are programmed in double precision, and the required accuracy of the numerical integration solution of Eq. (27) is set at 10^{-11}. Table I shows the numerical values (together with phase information) of the solutions obtained by the three different methods and Fig. 1 shows the difference between the numerically integrated solution and the solutions obtained by other methods. Since the solution value varies between 0.86 and 0.68, Fig. 1 practically represents the relative errors also. Fig. 2 shows, for various values of k in Eq. (27), the phase advance information with respect to the undamped oscillatory motion. In Fig. 1 the Poincaré perturbation solution is computed only up to $x = -10^{-4}$, since the first order solution is no longer valid for $x>-10^{-4}$ as seen from Eq. (32). We also note that the error of the first order perturbation solution indeed increases according to $\varepsilon^2 0(\frac{\ln x}{x})$. Fig. 1 indicates that the solutions of the present method are a few orders of magnitude more accurate than the first order perturbation solution as x approaches the apparent singular point. In fact the solution is nonsingular as x approaches zero from negative values. Errors in the region from $x = -10^{-1}$ to $x\to-0$ are practically uniform for both k=0 and k=1. When k=0 the solution is slightly too large; when k=1 it is slightly too small. For a particular value of k_e in between

0 and 1, the solution error changes sign at a point x_e (corresponding to k_e) where an exact solution is located as predicted by Eq. (25). Accordingly, there is an optimum value of k which minimizes errors in the region near x=0. By varying k, the optimum value appears to be near 0.8315 as seen from Fig. 1. At this value of k, the solution not only gives the most accurate absolute value but also the most correct phase advance information as seen from Fig. 2. In most physical problems, except in space trajectory studies, such an extreme accuracy is not necessary and there is no need to optimize the value of k. Accordingly it is recommended that one should set k=1 in Eq. (26) to obtain a simple yet very accurate analytic solution.

Having shown the extreme numerical accuracy of Eq. (29) as an approximate solution of the differential equation (27) in the region including the singular point, we ask the question how sensitive is the accuracy of the solution to the magnitude of the perturbing parameter ε? Again we cannot find an answer in analytic form. For each value of ε, however, the difference between the exact solution obtained by numerically integrating Eq. (27) and the approximate solution of Eq. (29) should give us some insight. For this comparison, we shall use the approximate solution in the form of Eq. (29) with k=1.

This is done for values of $\varepsilon = 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$ and 1, and is shown in Fig. 3 which is plotted from the computed results shown in Table II. From Fig. 3 we can conclude that the error of the approximate solution of Eq. (29) is roughly proportional to ε^2. It is also seen that for each value of ε the errors remain fairly constant as x approaches zero.

Turning our attention back to Eq. (27), we note that it has an irregular singularity at x=0 in the coefficient of y'. Accordingly,one of the solutions of Eq. (27) is analytic and the second solution has an essential singularity[3] at x=0. This is in fact what our approximate solution in the form of Eq. (29) shows for the case $\mu(x)= x^{-2}$. There is no singularity in the solution as x approaches zero from negative values as mentioned earlier, but the essential singularity in the solution appears as x approaches zero from positive values.

We have shown that Eq. (29) is an approximate solution of the differential equation in the form of Eq. (1), yet we cannot show the accuracy estimate of the approximate solution for the general case of $\mu(x)=x^{-n}$. For the case $\mu(x)=x^{-2}$, we have shown the approximate solution is accurate to $0(\varepsilon^2)$, as $x \to 0^-$ from negative values, by comparing it with the numerically integrated solution. To obtain the accuracy of the approximate solution for other cases, it would be necessary to have a similar numerical comparison.

Inspection of Eq. (26) indicates that $e^{-\varepsilon \int_{x_i}^{x} \mu dx}$ is the controlling factor in the integrand since the remaining factors in the equation are either constants or phase factors. As x approaches the singular point of $\mu(x)$, $e^{-\varepsilon \int_{x_i}^{x} \mu dx}$ will likewise approach zero or infinity, depending on (a) the sign of x in the region of interest, (b) the sign of $\varepsilon\mu(x)$, and (c) the fact whether $\mu(x)$ is even or odd. In general, we can state that when $e^{-\varepsilon \int_{x_i}^{x} \mu dx}$ approaches zero as x approaches the singular point the approximate solution approaches a constant. When $e^{-\varepsilon \int_{x_i}^{x} \mu dx}$ approaches infinity as x approaches the singular point, the approximate solution and its error both become singular.

In passing we mention that Eq. (1) with a forcing term added in fact can be derived from a linearized restricted three-body problem describing a body of negligible mass moving under the influence of the sun and aiming to impact a fictious point mass planet. In this case y represents the reciprocal of the distance between the sun and the spacecraft and x represents the angle between the sun-planet line and the sun-spacecraft line.

Physically it is quite clear that y must remain finite as $x \to 0$ from negative values and y does not have any physical meaning after the impact, i.e., mathematically Eq. (1) does not represent the particular physical problem in the positive x region.

Conclusions

It has been shown by heuristic arguments that a possible approximate solution to the differential equation (2) has the form of Eq. (26). Since the exact solution of Eq. (2) is not available, it is in general very difficult to obtain an analytic expression for the error estimate of the approximate solution. For the case $\mu(x) = x^{-2}$, however, we have shown that the approximate solution is accurate to $0(\varepsilon^2)$, as $x \to 0^-$ from negative values, by comparing it with the numerically integrated solution. The approximate solution is orders of magnitude more accurate than Poincaré's first order perturbation solution in the region near the singular point of $\mu(x)$.

The approximate analytic solution of the present method can yield significant mathematical and physical insight to actual problems.

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Attachments
Table I and II
References
Figures 1-3

TABLE I NUMERICAL COMPARISON OF SOLUTIONS

x	y_{NI}	y_P	$y_{k=0}$	$y_{k=1}$
-3×10^{-1}	0.863,209,366,648	0.863,209,366,648	0.863,209,366,649 $\angle 1.44^{o} \times 10^{-17}$	0.863,209,366,649 $\angle 2.88^{o} \times 10^{-17}$
-2×10^{-1}	0.808,500,274,368	0.808,500,274,562	0.808,500,274,383 $\angle 4.31^{o} \times 10^{-4}$	0.808,500,273,775 $\angle 9.31^{o} \times 10^{-4}$
-1×10^{-1}	0.745,730,231,672	0.745,730,236,130	0.745,730,232,162 $\angle 2.69^{o} \times 10^{-3}$	0.745,730,228,663 $\angle 2.69^{o} \times 10^{-3}$
-1×10^{-2}	0.682,963,384,873	0.682,963,644,168	0.682,963,395,159 $\angle 1.52^{o} \times 10^{-2}$	0.682,963,377,086 $\angle 1.52^{o} \times 10^{-2}$
-1×10^{-3}	0.676,522,149,188	0.676,525,553,550	0.676,522,174,303 $\angle 2.83^{o} \times 10^{-2}$	0.675,522,141,219 $\angle 2.86^{o} \times 10^{-2}$
-1×10^{-4}	0.676,000,129,830	0.676,031,393,989	0.676,000,165,951 $\angle 3.94^{o} \times 10^{-2}$	0.676,000,122,023 $\angle 4.22^{o} \times 10^{-2}$
-1×10^{-5}	0.675,989,168,251		0.675,989,206,029 $\angle 4.37^{o} \times 10^{-2}$	0.675,989,160,619 $\angle 7.23^{o} \times 10^{-2}$
-1×10^{-6}	0.675,989,168,220		0.675,989,206,000 $\angle 4.42^{o} \times 10^{-2}$	0.675,989,160,591 $\angle 3.31^{o} \times 10^{-1}$
-1×10^{-7}	0.675,989,168,219		0.675,989,205,999 $\angle 4.46^{o} \times 10^{-2}$	0.675,989,160,590 $\angle 2.98^{o}$

x	$y_{NI} - y_P$	$y_{NI} - y_{k=0}$	$y_{NI} - y_{k=1}$
-3×10^{-1}	0	-8.74×10^{-13}	8.74×10^{-13}
-2×10^{-1}	-1.94×10^{-10}	-1.51×10^{-11}	5.93×10^{-10}
-1×10^{-1}	-4.46×10^{-9}	-4.90×10^{-10}	3.01×10^{-9}
-1×10^{-2}	-2.59×10^{-7}	-1.03×10^{-8}	7.79×10^{-9}
-1×10^{-3}	-3.40×10^{-6}	-2.57×10^{-8}	7.97×10^{-9}
-1×10^{-4}	-3.13×10^{-5}	-3.61×10^{-8}	7.81×10^{-9}
-1×10^{-5}		-3.78×10^{-8}	7.63×10^{-9}
-1×10^{-6}		-3.78×10^{-8}	7.63×10^{-9}
-1×10^{-7}		-3.78×10^{-8}	7.63×10^{-9}

TABLE II NUMERICAL COMPARISON OF SOLUTIONS

x		$\epsilon = 10^{-4}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-1}$	$\epsilon = 1$
-0.3	y=	0.863,209,367	0.863,209,367	0.863,209,367	0.863,209,367	0.863,209,367
-0.2		0.808,500,274	0.808,535,032	0.808,875,614	0.811,604,511	0.726,263,869
-10^{-1}		0.745,730,229	0.745,954,706	0.748,134,341	0.764,482,024	0.708,624,475
-10^{-2}		0.682,963,377	0.684,381,156	0.696,577,684	0.747,712,320	0.705,580,025
-10^{-3}		0.676,522,141	0.679,156,911	0.695,413,136	0.747,712,268	"
-10^{-4}		0.676,000,122	0.679,046,496	0.695,413,133	"	"
-10^{-5}		0.675,989,161	"	"	"	"
-10^{-6}		0.675,989,161	"	"	"	"
-0.3	y_{NI}=	0.863,209,367	0.863,209,367	0.863,209,367	0.863,209,367	0.863,209,367
-0.2		0.808,500,274	0.808,535,092	0.808,881,507	0.812,176,793	0.833,280,389
-10^{-1}		0.745,730,232	0.745,955,006	0.748,163,625	0.766,848,989	0.827,964,623
-10^{-2}		0.682,963,385	0.684,381,924	0.696,644,755	0.750,960,965	0.827,626,744
-10^{-3}		0.676,522,149	0.679,157,691	0.695,478,970	0.750,957,898	0.827,626,464
-10^{-4}		0.676,000,130	0.679,047,258	0.695,478,939	0.750,957,896	"
-10^{-5}		0.675,989,168	"	"	"	"
-10^{-6}		0.675,989,168	"	"	"	"
-0.3	y_{NI}-y=	0	0	0	0	0
-0.2		0.059×10^{-8}	0.060×10^{-6}	0.059×10^{-4}	0.051×10^{-2}	0.107
-10^{-1}		0.301×10^{-8}	0.300×10^{-6}	0.293×10^{-4}	0.237×10^{-2}	0.119
-10^{-2}		0.778×10^{-8}	0.768×10^{-6}	0.671×10^{-4}	0.325×10^{-2}	"
-10^{-3}		0.797×10^{-8}	0.780×10^{-6}	0.658×10^{-4}	"	"
-10^{-4}		0.781×10^{-8}	0.762×10^{-6}	"	"	"
-10^{-5}		0.763×10^{-8}	0.761×10^{-6}	"	"	"
-10^{-6}		"	"	"	"	"

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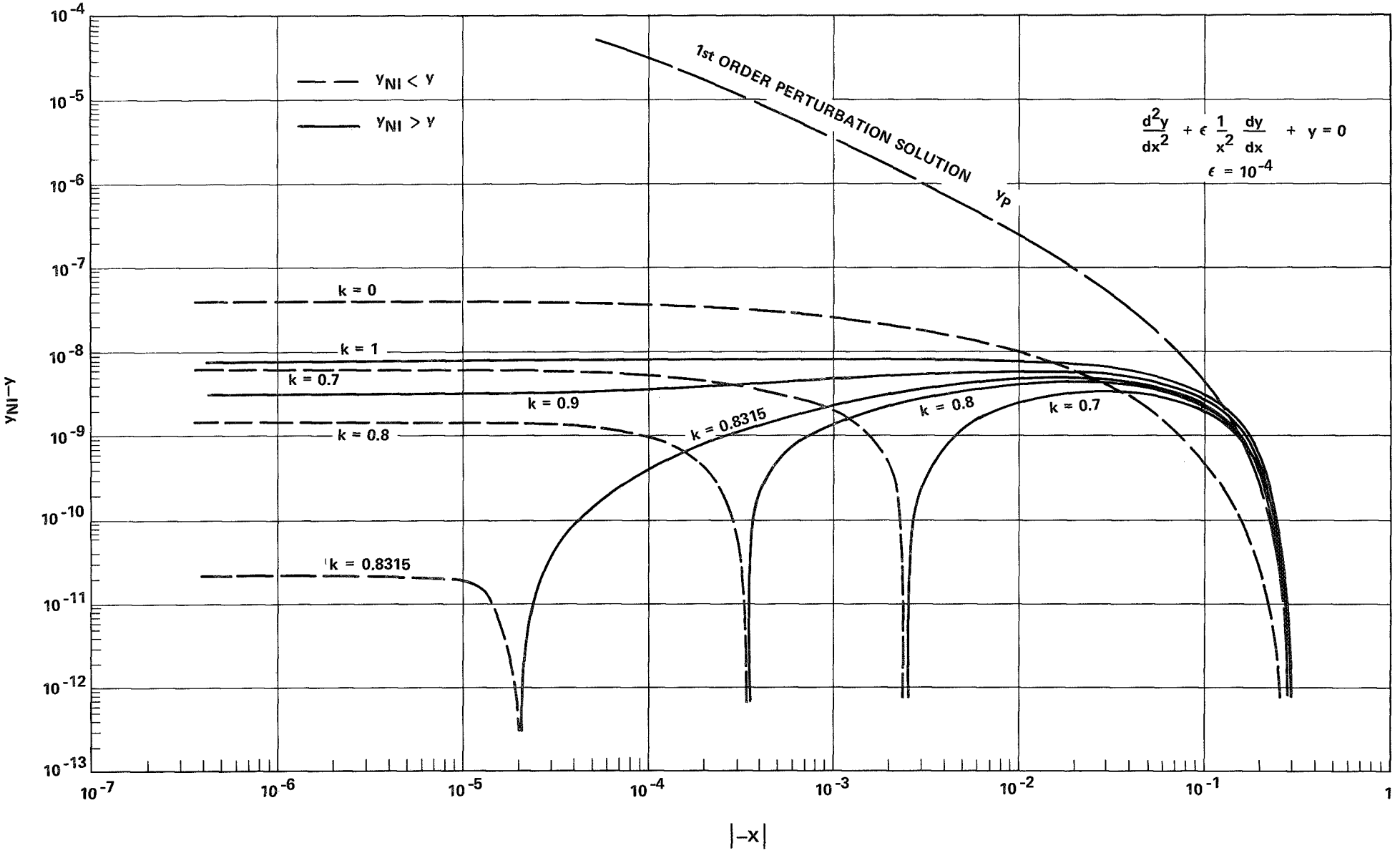

FIGURE 1 - COMPARISON OF SOLUTION MAGNITUDE DIFFERENCES.

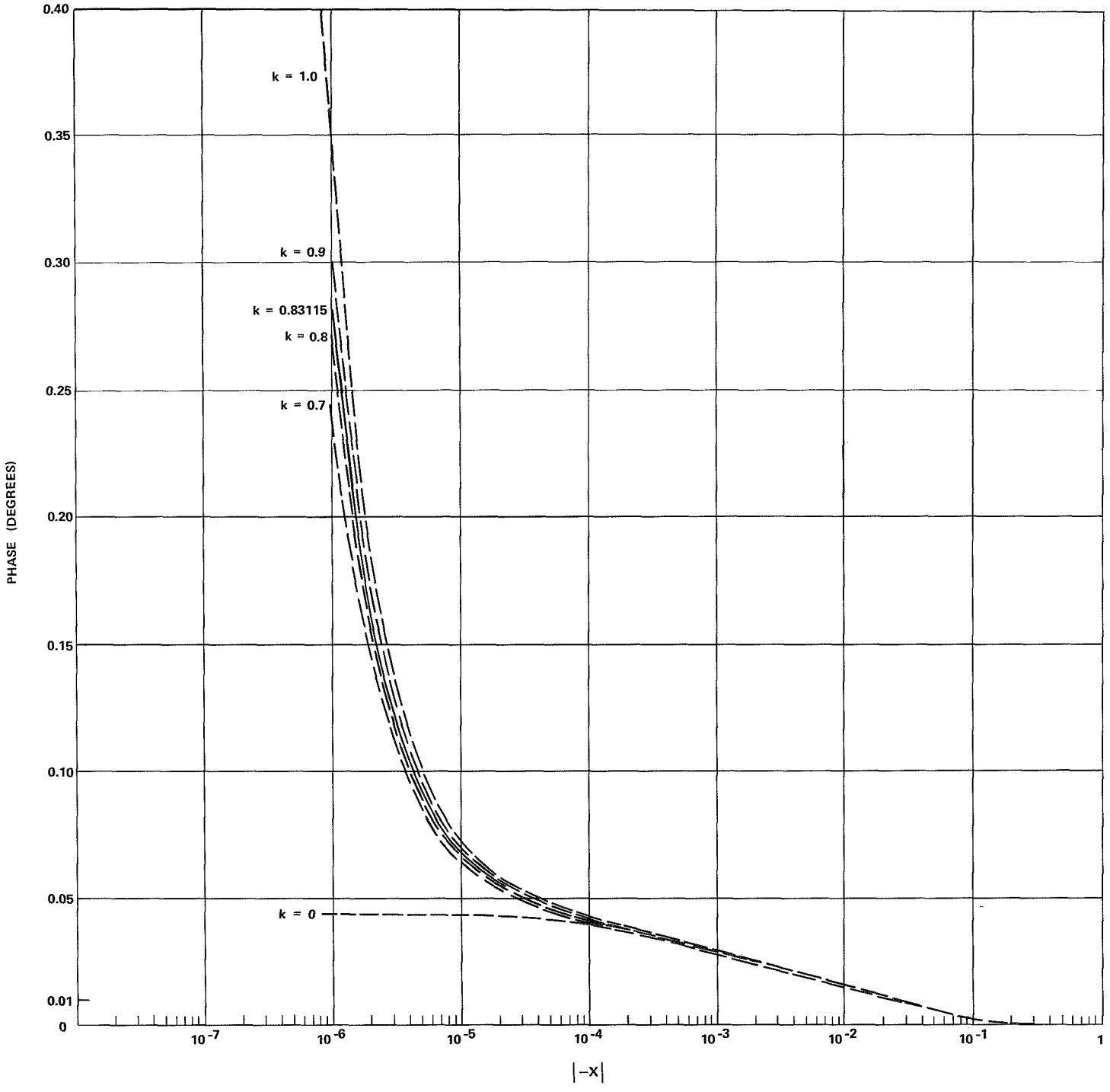

FIGURE 2 - PHASE INFORMATION.

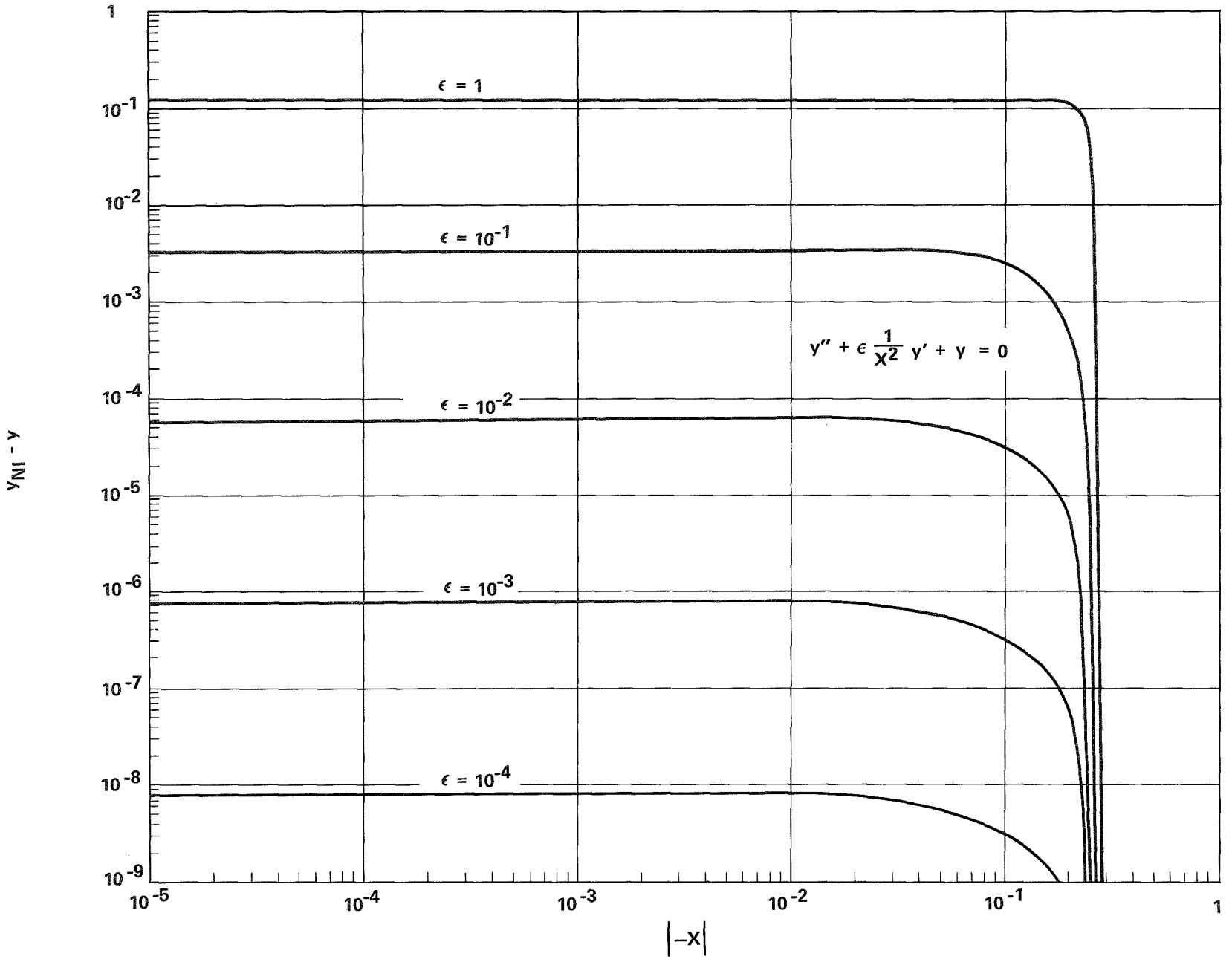

FIGURE 3 - EFFECT OF ϵ ON THE ACCURACY OF THE APPROXIMATE SOLUTION